

Mathematics 2015 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. Write the value of $\Delta = \begin{vmatrix} x+y & x+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$ **[1]**

Solution : Given, $\Delta = \begin{vmatrix} x+y & x+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2$, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$$

$$\Delta = -3(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

[Taking out $(x+y+z)$ and (-3) common from R_1 and R_3 respectively]

$$\Delta = -3(x+y+z).\,0$$

[$\because$ R_1 and R_3 are identical]

$\Rightarrow \quad \Delta = 0$ **Ans.**

2. Write the sum of the order and degree of the following differential equation :

$$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0 \qquad \textbf{[1]}$$

Solution : Given,

$$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0.$$

On differentiating w.r.t. x both sides, we get

$$3\left(\frac{dy}{dx}\right)^2 \times \frac{d^2y}{dx^2} = 0.$$

Since the order and degree of the differential equation is 2 and 1 respectively.

So, the sum of the order and degree is 3. **Ans.**

3. Write the integrating factor of the following differential equation :

$$(1+y^2)+(2xy-\cot y)\frac{dy}{dx} = 0. \qquad \textbf{[1]}$$

Solution : Given,

$$(1+y^2)+(2xy-\cot y)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1+y^2)\,dx + 2xy\,dy = \cot y\,dy$$

$$\Rightarrow \quad \frac{dx}{dy}+\left(\frac{2y}{1+y^2}\right)x = \frac{\cot y}{1+y^2}$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S,$$

where $\quad R = \dfrac{2y}{1+y^2}$

and $\quad S = \dfrac{\cot y}{1+y^2}$

$\therefore$ Integrating factor $= e^{\int R\,dy} = e^{\int \frac{2y}{1+y^2}dy}$

$= e^{\log(1+y^2)}$

$= 1+y^2.$ **Ans.**

4. If $\hat{a}, \hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, then find the value of $|2\hat{a}+\hat{b}+\hat{c}|$. **[1]**

Solution : Let $x = |2\hat{a}+\hat{b}+\hat{c}|$

$\therefore \quad x^2 = |2\hat{a}+\hat{b}+\hat{c}|^2$

$= 4|\hat{a}|^2+|\hat{b}|^2+|\hat{c}|^2+ 2(2\hat{a}.\hat{b}+\hat{b}.\hat{c}+2\hat{c}.\hat{a})$

Since, $\hat{a}, \hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors.

$\therefore \quad \hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = 0$

So, $\quad x^2 = 4\times 1+1+1+2\,(2\times 0+0+2\times 0)$

$\Rightarrow \quad x^2 = 6$

$\therefore \quad x = \sqrt{6}$

Hence, $|2\hat{a}+\hat{b}+\hat{c}| = \sqrt{6}.$ **Ans.**

5. Write a unit vector perpendicular to both the vectors $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}$. **[1]**

Solution : A vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\vec{a}\times\vec{b}$

$\therefore$ Unit vector $\perp$ to $\vec{a}$ and $\vec{b}$ $= \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \hat{i}(0-1) - \hat{j}(0-1) + \hat{k}(1-1)$$

$$= -\hat{i} + \hat{j}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$\therefore$ Required unit vector $= \dfrac{-\hat{i}+\hat{j}}{\sqrt{2}}$ **Ans.**

6. The equations of a line are $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line. [1]

Solution : Given line is $5x - 3 = 15y + 7 = 3 - 10z$

Rewritting the eq. in standard form :

$$5\left(x - \frac{3}{5}\right) = 15\left(y + \frac{7}{15}\right) = -10\left(z - \frac{3}{10}\right)$$

i.e., $$\frac{x - \frac{3}{5}}{\frac{1}{5}} = \frac{y + \frac{7}{15}}{\frac{1}{15}} = \frac{z - \frac{3}{10}}{\frac{-1}{10}}$$

Thus, the direction ratios of the line are

$\frac{1}{5}, \frac{1}{15}, \frac{-1}{10}$ *i.e.,* $6, 2, -3$.

Hence, its direction cosines are

$$\pm\frac{6}{\sqrt{6^2+2^2+(-3)^2}}, \pm\frac{2}{\sqrt{6^2+2^2+(-3)^2}},$$

$$\pm\frac{-3}{\sqrt{6^2+2^2+(-3)^2}} \text{ i.e., } \pm\frac{6}{7}, \pm\frac{2}{7}, \mp\frac{3}{7}$$

i.e., $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$ or $\frac{-6}{7}, \frac{-2}{7}, \frac{3}{7}$. **Ans.**

SECTION — B

7. To promote the making of toilets for women, an organisation tried to generate awareness through

(i) house calls (ii) letters, and (iii) announcements. The cost for each mode per attempt is given below :

(i) ₹ 50 (ii) ₹ 20 (iii) ₹ 40

The number of attempts made in three villages X, Y and Z are given below :

	(i)	(ii)	(iii)
X	400	300	100
Y	300	250	75
Z	500	400	150

Find the total cost incurred by the organisation for the three villages separately, using matrices.

Write one value generated by the organisation in the society. [4]

Solution : The number of attempts made in three villages X, Y and Z can be represented by the 3 × 3 matrix.

$$X = \begin{bmatrix} 400 & 300 & 100 \\ 300 & 250 & 75 \\ 500 & 400 & 150 \end{bmatrix}$$

and the cost for each mode per attempt can be represented by the 3 × 1 matrix.

$$Y = \begin{bmatrix} 50 \\ 20 \\ 40 \end{bmatrix}$$

$\therefore$ By matrix multiplication the cost incurred by the organisation for the three villages.

$$XY = \begin{bmatrix} 400 & 300 & 100 \\ 300 & 250 & 75 \\ 500 & 400 & 150 \end{bmatrix}\begin{bmatrix} 50 \\ 20 \\ 40 \end{bmatrix}$$

$$\Rightarrow \quad XY = \begin{bmatrix} 30,000 \\ 23,000 \\ 39,000 \end{bmatrix}$$

Hence the total cost incurred by the organisation for the three villages separately are ₹ 30,000, ₹ 23,000 and ₹ 39,000.

The organisation in the society generated the value of cleanliness for the women welfare. **Ans.**

8. Solve for x :

$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$ [4]

Solution : Given, $\tan^{-1}(x+1) + \tan^{-1}(x-1)$

$$= \tan^{-1}\frac{8}{31}$$

$$\Rightarrow \quad \tan^{-1}\frac{x+1+x-1}{1-(x+1)(x-1)} = \tan^{-1}\frac{8}{31}$$

$$\left[\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}\right]$$

$$\Rightarrow \quad \tan^{-1}\frac{2x}{2-x^2} = \tan^{-1}\frac{8}{31}$$

$$\Rightarrow \quad \frac{2x}{2-x^2} = \frac{8}{31}$$

$\Rightarrow \quad 62x = 16 - 8x^2$

$\Rightarrow \quad 4x^2 + 31x - 8 = 0$

Solving the quadratic equation, we get

$$(4x - 1)(x + 8) = 0$$

$\Rightarrow \quad x = \frac{1}{4}$ or -8.

Since, $x = -8$ doesn't satisfy the given equation. So neglecting it.

$\therefore \quad x = \frac{1}{4}$. **Ans.**

OR

Prove the following :

$$\cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right) = 0 \ (0 < xy, yx, zx < 1).$$

Solution : L. H. S.

$$= \cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right)$$

$$= \tan^{-1}\left(\frac{x-y}{xy+1}\right)+\tan^{-1}\left(\frac{y-z}{yz+1}\right)+\tan^{-1}\left(\frac{z-x}{zx+1}\right)$$

$$\left[\because \tan^{-1} x = \cot^{-1}\frac{1}{x}\right]$$

$= \tan^{-1}x - \tan^{-1}y + \tan^{-1}y - \tan^{-1}z + \tan^{-1}z - \tan^{-1}x = 0 =$ R. H. S. **Hence Proved.**

9. Using properties of determinants, prove the following:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2. \quad \textbf{[4]}$$

Solution :

$$\text{Let } \Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking a, b and c common from C_1, C_2 and C_3 respectively.

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = abc\begin{vmatrix} 2(a+b) & 2(b+c) & 2(a+c) \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Taking 2 common from R_1,

$$\Delta = 2abc\begin{vmatrix} a+b & b+c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_1$

$$\Delta = 2abc\begin{vmatrix} 0 & c & c \\ a+b & b & a \\ -a & 0 & -a \end{vmatrix}$$

Now, taking c and a common from R_1 and R_3

$$\Delta = 2a^2bc^2\begin{vmatrix} 0 & 1 & 1 \\ a+b & b & a \\ -1 & 0 & -1 \end{vmatrix}$$

Expanding along R_1,

$\Delta = 2a^2bc^2\,[0(-b-0) -1\{-(a+b)+a\} + 1(0+b)]$

$= 2a^2bc^2\,[0-(-a-b+a)+b]$

$= 2a^2bc^2\,[0+b+b]$

$= 2a^2bc^2\,[2b]$

$= 4a^2b^2c^2 =$ R. H. S. **Hence Proved.**

10. Find the adjoint of the matrix :

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

and hence show that A. (adj A) = |A| I_3. [4]

Solution : We have,

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$A_{11} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 1 - 4 = -3$$

$$A_{12} = -\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -(2+4) = -6$$

$$A_{13} = \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -4 - 2 = -6$$

$$A_{21} = -\begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} = -(-2-4) = 6$$

$$A_{22} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1 + 4 = 3$$

$$A_{23} = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$$

$$A_{31} = \begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix} = +4+2 = +6$$

$$A_{32} = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$$

$$A_{33} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1+4 = +3$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= -1(1-4)+2(2+4)-2(-4-2)$$

$$= -1(-3)+2(6)-2(-6)$$

$$= 3+12+12 = 27$$

Now L. H. S. = A (adj A)

$$= \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix}$$

$$= 27\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= |A|.I_3 =$ R. H. S. **Hence Proved.**

11. Show that the function $f(x) = |x-1| + |x+1|$, for all $x \in R$, is not differentiable at the points $x = -1$ and $x = 1$. [4]

Solution : Given,

$$f(x) = |x-1| + |x+1|$$

$$= \begin{cases} -(x-1)-(x+1) = -2x, & x < -1 \\ -(x-1)+(x+1) = 2, & -1 \le x < 1 \\ (x-1)+(x+1) = 2x, & x \ge 1 \end{cases}$$

Now differentiability at $x = -1$

$$\text{LHD} = \lim_{h\to 0} \frac{f(-1-h)-f(-1)}{-h}$$

$$= \lim_{h\to 0} \frac{-2(-1-h)-2}{-h}$$

$$= \lim_{h\to 0} \frac{2+2h-2}{-h}$$

$$= \lim_{h\to 0} \frac{2h}{-h}$$

$$= \lim_{h\to 0} -2$$

$$= -2$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$$

$$= \lim_{h\to 0} \frac{2-2}{h} = \lim_{h\to 0} 0$$

$$= 0$$

Since (LHD) $\neq$ (RHD)

$\therefore f(x)$ is not differentiable at $x = -1$.

Now differentiability at $x = 1$

$$(\text{LHD at } x = 1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{2-2}{-h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

$$(\text{RHD at } x = 1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{2(1+h)-2}{h}$$

$$= \lim_{h\to 0} \frac{2+2h-2}{h}$$

$$= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2$$

$$= 2$$

Since LHD $\neq$ RHD

$\therefore f(x)$ is not differentiable at $x = 1$, also.

Hence Proved.

12. If $y = e^{m\sin^{-1}x}$, then show that

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0. \quad \textbf{[4]}$$

Solution : Given,

$$y = e^{m\sin^{-1}x}, \quad ...(i)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{m\sin^{-1}x} \times \frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = m e^{m\sin^{-1}x}$$

Again differentiating, w.r.t. x, we get

$$\sqrt{1-x^2}\frac{d^2y}{dx^2}+\frac{dy}{dx}\times\left(\frac{-x}{\sqrt{1-x^2}}\right) = \frac{m^2 e^{m\sin^{-1}x}}{\sqrt{1-x^2}}$$

Multiplying both sides by $\sqrt{1-x^2}$, we get

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx} = m^2 e^{m\sin^{-1}x}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y = 0 \quad \text{[using (i)]}$$

Hence Proved.

13. **If $f(x)=\sqrt{x^2+1}$; $g(x)=\dfrac{x+1}{x^2+1}$ and $h(x)=2x-3$, then find $f'[h'\{g'(x)\}]$.** **[4]**

Solution : Given,

$$f(x) = \sqrt{x^2+1},\ g(x) = \frac{x+1}{x^2+1}$$

and $h(x) = 2x-3$

Now, $$f'(x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

$$g'(x) = \frac{(x^2+1)-(x+1)2x}{(x^2+1)^2} = \frac{1-2x-x^2}{(x^2+1)^2}$$

and $h'(x) = 2$.

$$\therefore f'[h'\{g'(x)\}] = f'\left[h'\left\{\frac{1-2x-x^2}{(x^2+1)^2}\right\}\right]$$

$$= f'[2] \qquad [\because h'(x)=2]$$

$$= \frac{2}{\sqrt{2^2+1}} = \frac{2}{\sqrt{5}}.$$ **Ans.**

14. **Evaluate : $\int(3-2x).\sqrt{2+x-x^2}\,dx$.** **[4]**

Solution :

Let, $I = \int(3-2x).\sqrt{2+x-x^2}\,dx.$

Let $3-2x = \lambda\frac{d}{dx}(2+x-x^2)+\mu$

$\Rightarrow \quad 3-2x = \lambda(1-2x)+\mu \qquad$ ...(i)

$\Rightarrow \quad 3-2x = (-2\lambda)x+\lambda+\mu$

Equating the coefficients of like terms, we get

$-2\lambda = -2$

$\Rightarrow \quad \lambda = 1$

and $\quad \lambda+\mu = 3$

$\Rightarrow \quad \mu = 3-1 = 2$

$\therefore$ Equation. (i) becomes $3-2x = (1-2x)+2$

$$I = \int\{(1-2x)+2\}\sqrt{2+x-x^2}\,dx$$

$$\Rightarrow I = \int(1-2x)\sqrt{2+x-x^2}\,dx+2\int\sqrt{2+x-x^2}\,dx$$

$$\Rightarrow I = I_1+2I_2 \qquad \text{...(ii)}$$

$$\Rightarrow I_1 = \int(1-2x)\sqrt{2+x-x^2}\,dx$$

Let $2+x-x^2 = t$

$\Rightarrow \quad (1-2x)\,dx = dt$

$$\therefore \quad I_1 = \int\sqrt{t}\,dt$$

$$\Rightarrow \quad I_1 = \frac{t^{1/2+1}}{\frac{1}{2}+1}+C_1$$

$$\Rightarrow \quad I_1 = \frac{2}{3}t^{3/2}+C_1$$

$$\Rightarrow \quad I_1 = \frac{2}{3}(2+x-x^2)^{3/2}+C_1 \qquad \text{...(iii)}$$

and $$I_2 = \int\sqrt{2+x-x^2}\,dx$$

$$\Rightarrow \quad I_2 = \int\sqrt{2+\frac{1}{4}-\left(x^2-x+\frac{1}{4}\right)}dx$$

$$\Rightarrow \quad I_2 = \int\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$\Rightarrow \quad I_2 = \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}+\frac{\frac{9}{4}}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)+C_2$$

$$\Rightarrow \quad I_2 = \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{2+x-x^2}+\frac{9}{8}\sin^{-1}\left(\frac{2x-1}{3}\right)+C_2 \qquad \text{...(iv)}$$

From (ii), (iii) and (iv), we get

$$I = \frac{2}{3}(2+x-x^2)^{3/2}+\left(x-\frac{1}{2}\right)\sqrt{2+x-x^2}+\frac{9}{4}\sin^{-1}\left(\frac{2x-1}{3}\right)+C,$$

where $\quad C = C_1+C_2.$ **Ans.**

OR

Evaluate : $\int\dfrac{x^2+x+1}{(x^2+1)(x+2)}dx.$

Solution : Let $\quad I = \int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$

$$I = \int \left(\frac{x^2+1}{(x^2+1)(x+2)} + \frac{x}{(x^2+1)(x+2)}\right)dx$$

$$\Rightarrow \quad I = \int \frac{1}{x+2}dx + \int \frac{x}{(x^2+1)(x+2)}dx$$

$$\Rightarrow \quad I = \log|x+2| + \int \frac{x}{(x^2+1)(x+2)}dx \quad \text{...(i)}$$

Let $\quad \frac{x}{(x^2+1)(x+2)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$

$$x = A(x^2+1) + (Bx+C)(x+2) \quad \text{...(ii)}$$

Putting $x+2=0$

$\Rightarrow \quad x = -2$ in (ii), we get $-2 = 5A$

$$\Rightarrow \quad A = \frac{-2}{5}$$

Putting $\quad x = 0$ and $x = 1$ in (ii), we get

$$0 = A + 2C$$

$$\Rightarrow \quad C = \frac{1}{5}$$

$$1 = 2A + 3B + 3C$$

$$\Rightarrow \quad 3B = \frac{6}{5}$$

$$\Rightarrow \quad B = \frac{2}{5}$$

$$\therefore \quad I = \log|x+2| - \frac{2}{5}\int \frac{dx}{x+2} + \int \frac{\frac{2}{5}x + \frac{1}{5}}{x^2+1}dx$$

$$\Rightarrow \quad I = \log|x+2| - \frac{2}{5}\int \frac{dx}{x+2} + \frac{2}{5}\int \frac{xdx}{x^2+1} + \frac{1}{5}\int \frac{1}{x^2+1}dx$$

$$\Rightarrow \quad I = \log|x+2| - \frac{2}{5}\log|x+2| + \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + C$$

$$\therefore \quad I = \frac{3}{5}\log|x+2| + \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + C. \quad \textbf{Ans.}$$

15. Find $\left(\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}.\right)$. **[4]**

Solution : Let $\quad I = \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$.

Multiplying and dividing denominator by $\cos x$

$$\int_0^{\pi/4} \frac{dx}{\cos^4 x \frac{\sqrt{2\sin 2x}}{\cos x}} = \int_0^{\pi/4} \frac{\sec^4 x\, dx}{\sqrt{\frac{2\sin 2x}{\cos^2 x}}}$$

$$I = \int_0^{\pi/4} \frac{(1+\tan^2 x)\sec^2 x}{2\sqrt{\frac{\sin x\cos x}{\cos^2 x}}}dx$$

$$[\because \sin 2x = 2\sin x\cos x]$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{(1+\tan^2 x)\sec^2 x}{2\sqrt{\tan x}}dx$$

Putting $\quad \tan x = t$

$\Rightarrow \quad \sec^2 x\, dx = dt$, we get

$$\left(\begin{array}{l}\because \text{when } x=0, t=0 \\ \text{and } \quad x=\frac{\pi}{4}, t=1\end{array}\right)$$

$$\therefore \quad I = \int_0^1 \frac{(1+t^2)dt}{2\sqrt{t}}$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1 \left(\frac{1}{\sqrt{t}} + t^{\frac{3}{2}}\right)dt$$

$$\Rightarrow \quad I = \frac{1}{2}\left[2\sqrt{t} + \frac{2}{5}t^{\frac{5}{2}}\right]_0^1$$

$$\Rightarrow \quad I = \left[\sqrt{t} + \frac{1}{5}t^{\frac{5}{2}}\right]_0^1$$

$$\Rightarrow \quad I = \left[\sqrt{1} + \frac{1}{5}1^{\frac{5}{2}}\right] - \left[\sqrt{0} + \frac{1}{5}0^{\frac{5}{2}}\right]$$

$$\therefore \quad I = \frac{6}{5}. \quad \textbf{Ans.}$$

16. Find $\left(\int \frac{\log x}{(x+1)^2}dx.\right)$. **[4]**

Solution : Let $\quad I = \int \frac{\log x}{(x+1)^2}dx$

$$= \int \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{\frac{1}{(x+1)^2}}dx$$

Integrating by parts

$$I = \log x \int \frac{dx}{(x+1)^2} - \int\left(\frac{d}{dx}\log x \int \frac{dx}{(x+1)^2}\right)dx$$

$$\Rightarrow I = -\frac{\log x}{(x+1)} + \int \frac{1}{x(x+1)}dx \quad \left[\because \frac{d}{dx}\frac{1}{x^2} = -\frac{1}{x}\right]$$

$$= -\frac{\log x}{(x+1)} + \int \frac{(x+1)-x}{x(x+1)} dx$$

[Add and subtract x]

$$\Rightarrow \quad I = -\frac{\log x}{(x+1)} + \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$\Rightarrow \quad I = -\frac{\log x}{(x+1)} + \log|x| - \log|x+1| + C$$

$$\therefore \quad I = \log\left|\frac{x}{x+1}\right| - \frac{\log x}{(x+1)} + C. \qquad \textbf{Ans.}$$

17. If $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 5\hat{k}$ then find a unit vector perpendicular to both of the vectors $(\vec{a} - \vec{b})$ and $(\vec{c} - \vec{b})$. **[4]**

Solution : Here,

$$(\vec{a} - \vec{b}) = \hat{i} + 2\hat{j} + \hat{k} - 2\hat{i} - \hat{j}$$
$$= -\hat{i} + \hat{j} + \hat{k}$$

and
$$(\vec{c} - \vec{b}) = 3\hat{i} - 4\hat{j} - 5\hat{k} - 2\hat{i} - \hat{j}$$
$$= \hat{i} - 5\hat{j} - 5\hat{k}$$

So, required unit vector $\bar{r} = \dfrac{(\vec{a}-\vec{b}) \times (\vec{c}-\vec{b})}{\left|(\vec{a}-\vec{b}) \times (\vec{c}-\vec{b})\right|}$,

where

$$\vec{r} = (\hat{a} - \hat{b}) \times (\vec{c} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -5 & -5 \end{vmatrix}$$

$$= \hat{i}(-5+5) - \hat{j}(5-1) + \hat{k}\ (5-1)$$
$$= -4\hat{j} + 4\hat{k}$$

Hence, $\hat{r} = \dfrac{4\hat{k} - 4\hat{j}}{\sqrt{4^2 + 4^2}} = \left(\dfrac{\hat{k} - \hat{j}}{\sqrt{2}}\right)$. **Ans.**

18. Find the equation of a line passing through the point (1, 2, – 4) and perpendicular to two lines

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu\ (3\hat{i} + 8\hat{j} - 5\hat{k})$.[4]

Solution : Let the direction ratios of required line be a, b, c, since, the line is perpendicular to

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu\ (3\hat{i} + 8\hat{j} - 5\hat{k})$

$\therefore \quad 3a - 16b + 7c = 0$

and $\quad 3a + 8b - 5c = 0.$

Solving by cross multiplication, we get

$$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$$

$$\Rightarrow \quad \frac{a}{24} = \frac{b}{36} = \frac{c}{72}$$

$\therefore$ the direction ratios of line : 2, 3, 6.

Hence, required line through the point (1, 2, – 4) is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda\ (2\hat{i} + 3\hat{j} + 6\hat{k}) \ \textbf{Ans.}$$

OR

Find the equation of the plane passing through the points (–1, 2, 0), (2, 2, –1) and parallel to the line $\dfrac{x-1}{1} = \dfrac{2y+1}{2} = \dfrac{z+1}{-1}$.

Solution : The equation of a plane passing through (–1, 2, 0) is

$$a(x+1) + b(y-2) + c(z-0) = 0 \qquad \text{...(i)}$$

It passes through (2, 2, –1)

$$\therefore \quad a(2+1) + b(2-2) + c(-1-0) = 0$$
$$3a + 0b - c = 0 \qquad \text{...(ii)}$$

The given line is

$$\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$$

i. e.
$$\frac{x-1}{1} = \frac{y+\frac{1}{2}}{1} = \frac{z+1}{-1}$$

$\therefore$ *d. r'. s* of line are 1, 1, –1

The plane (i) is parallel to the given line

$$a + b - c = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$$

i.e., direction ratios of normal to the plane are 1, 2, 3.

$$\therefore \quad 1(x+1) + 2(y-2) + 3(z-0) = 0$$

i.e., $\quad x + 2y + 3z = 3.$ **Ans.**

19. Three cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spades. Hence find the mean of the distribution. **[4]**

Solution : Let X denote the number of spades when three cards are drawn, then, X is a random variable that can take values 0, 1, 2, 3.

Let E be the event when spade card is drawn,

$$p = P(E) = \frac{13}{52} = \frac{1}{4}$$

$$q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$$

P(X = 0) = Probability of getting no spade

$$= {}^3C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

P(X = 1) = Probability of getting one spade

$$= {}^3C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{3-1} = \frac{27}{64}$$

P(X = 2) = Probability of getting two spades

$$= {}^3C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{3-2} = \frac{9}{64}$$

P(X = 3) = Probability of getting three spades

$$= {}^3C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{3-3} = \frac{1}{64}$$

Thus, the probability distribution of random variable X is given by

X	0	1	2	3
P(X)	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

$\therefore$ Mean $(\overline{X}) = \Sigma XP(X)$

$$= 0\times\frac{27}{64} + 1\times\frac{27}{64} + 2\times\frac{9}{64} + 3\times\frac{1}{64}$$

$$= \frac{48}{64} = \frac{3}{4}.$$ **Ans.**

OR

For 6 trials of an experiment, let X be a binomial variate which satisfies the relation 9P(X = 4) = P(X = 2). Find the probability of success.

Solution : Let p denote the probability of getting success and q be the probability of failure.

Since, $P(x = r) = {}^nC_r\, p^r q^{n-r}$

$\therefore$ $P(x = 4) = {}^6C_4\, p^4 q^{6-4}$

and $P(x = 2) = {}^6C_2\, p^2 q^{6-2}$

We have $9P(X = 4) = P(X = 2)$

$\Rightarrow$ $9\,{}^6C_4\, p^4 q^{6-4} = {}^6C_2 p^2 q^4$ $[\because {}^6C_4 = {}^6C_2]$

$\Rightarrow$ $9p^2 = q^2$

$\Rightarrow$ $9p^2 = (1-p)^2$ $[\because p + q = 1]$

$\Rightarrow$ $9p^2 = 1^2 + p^2 - 2p$

$\Rightarrow$ $9p^2 - p^2 + 2p - 1 = 0$

$\Rightarrow$ $8p^2 + 2p - 1 = 0$

$\Rightarrow$ $8p^2 + 4p - 2p - 1 = 0$

$\Rightarrow$ $(4p - 1)(2p + 1) = 0$

$\therefore$ $p = \frac{1}{4}, \frac{-1}{2}$

Since probability can not be – ve

$\therefore$ $p = \frac{1}{4}$

Hence, the probability of success $= \frac{1}{4}$. **Ans.**

SECTION — C

20. Consider $f : R_+ \to [-9, \infty)$ given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with

$$f^{-1}(y) = \left(\frac{\sqrt{54+5y}-3}{5}\right).$$ **[6]**

Solution : To prove f is invertible we have to prove that f is one-one and onto.

For one-one

Let $x_1, x_2 \in R_+$, then

$$f(x_1) = f(x_2)$$

$\Rightarrow$ $5x_1^2 + 6x_1 - 9 = 5x_2^2 + 6x_2 - 9$

$\Rightarrow$ $5(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow$ $(x_1 - x_2)(5x_1 + 5x_2 + 6) = 0$

$\Rightarrow$ $x_1 - x_2 = 0$ as $5x_1 + 5x_2 + 6 \neq 0$

$\Rightarrow$ $x_1 = x_2$

i.e., f is one-one function.

For onto

Let $f(x) = y$

$\because$ $y = 5x^2 + 6x - 9$

$\therefore$ $5x^2 + 6x - (9 + y) = 0$

$$x = \frac{-6 \pm \sqrt{36 + 4\times5(9+y)}}{10}$$

$$= \frac{-6 \pm \sqrt{216 + 20y}}{10}$$

$$= \frac{\pm\sqrt{54+5y}-3}{5}$$

$\Rightarrow$ $x = \frac{\sqrt{54+5y}-3}{5}$ $(\because x \in R_+)$

Clearly $\forall\, y \in [-9, \infty]$, the value of $x \in R_+$

$\Rightarrow f$ is onto function.

Hence f is one-one onto function.

$\Rightarrow f$ is invertible function with

$$f^{-1}(y) = \frac{\sqrt{54+5y}-3}{5}$$ **Hence Proved.**

OR

A binary operation * is defined on the set $x = R - \{-1\}$ by $x * y = x + y + xy, \forall x, y \in X$. Check whether * is commutative and associative. Find its identity element and also find the inverse of each element of X.**

21. Find the value of p for which the curves $x^2 = 9p(9 - y)$ and $x^2 = p(y + 1)$ cut each other at right angles. **[6]**

Solution : Given,

$$x^2 = 9p\,(9 - y) \qquad ...(i)$$

and $$x^2 = p\,(y + 1) \qquad ...(ii)$$

From (i) and (ii), we get

$$9p\,(9 - y) = p(y + 1)$$

$$\Rightarrow \quad 81p - 9py = py + p$$

$$\Rightarrow \quad 10py = 80\,p$$

$$\Rightarrow \quad y = 8$$

$$\therefore \quad x^2 = 9p$$

Now, differentiating (i) and (ii) w.r.t. x, we get

$$2x = -9p\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{9p}$$

and $$2x = p\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{p}.$$

Since curves cut each other at right angles

$$\therefore \quad \left(\frac{2x}{p}\right)\left(-\frac{2x}{9p}\right) = -1 \qquad [\because m_1m_2 = -1]$$

$$\Rightarrow \quad \frac{-4x^2}{9p^2} = -1$$

$$\Rightarrow \quad \frac{4}{9p^2}(9p) = 1 \qquad (\because x^2 = 9p)$$

$$\therefore \quad p = 4.$$ **Ans.**

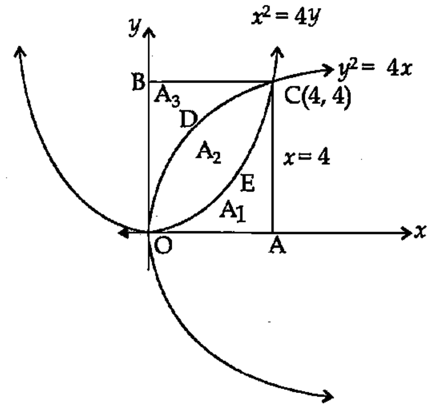
22. Using integration, prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 0$ and $y = 4$ into three equal parts. **[6]**

Solution : Given,

$$y^2 = 4x \qquad ...(i)$$

and $$x^2 = 4y \qquad ...(ii)$$

Solving (i) and (ii), we get the point of intersection (0, 0) and (4, 4).



The area of the region OECDO bounded by the given curve A_2 = Area under ($y^2 = 4x$) – Area under ($x^2 = 4y$)

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 = \left(\frac{32}{3} - \frac{16}{3}\right)$$

$$= \frac{16}{3} \text{ sq. units} \qquad ...(iii)$$

and the area of the region OACEO A_1 = Area under $x^2 = 4y$

$$= \int_0^4 \left(\frac{x^2}{4}\right)dx = \left[\frac{x^3}{12}\right]_0^4$$

$$= \frac{16}{3} \text{ sq. units} \qquad ...(iv)$$

Similarly, the area of the region ODCBO

$$A_3 = \text{Area of square} - (A_1 + A_2)$$

$$= 16 - \left(\frac{16}{3} + \frac{16}{3}\right)$$

$$= 16 - \frac{32}{3}$$

$$= \frac{16}{3} \text{ sq. units} \qquad ...(v)$$

From (iii), (iv) and (v), it can be concluded that the given curves divide the area of the square bounded by $x = 0$, $x = 4$, $y = 0$ into three equal parts. **Hence Proved.**

23. Show that the differential equation $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ is homogeneous and also solve it. **[6]**

Solution : We have

**Answer is not given due to the change in present syllabus

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2} \quad ...(i)$$

Let $f(x, y) = \frac{y^2}{xy - x^2}$

$\therefore$ $f(\lambda x, \lambda y) = \frac{\lambda^2 y^2}{\lambda^2 xy - \lambda^2 x^2} = \frac{y^2}{xy - x^2}$

$= \lambda^0 f(x, y).$

Hence, the differential equation is homogeneous.

Putting $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + \frac{xdv}{dx} = \frac{v^2x^2}{vx^2 - x^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{v-1}$$

$$\Rightarrow \quad \frac{v-1}{v}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \int\frac{v-1}{v}dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad \int\left(1 - \frac{1}{v}\right)dv = \int\frac{dx}{x}$$

$$v - \log|v| = \log|x| + C$$

$$\Rightarrow \frac{y}{x} - \log|y| + \log|x| = \log|x| + C$$

Hence, $x = \frac{y}{\log|y| + C}$ is the required solution of the differential equation. **Hence Proved.**

OR

Find the particular solution of the differential equation $(\tan^{-1}y - x)\,dy = (1 + y^2)\,dx$, given that $x = 1$, when $y = 0$.

Solution : Given,

$$(\tan^{-1} y - x)dy = (1 + y^2)dx$$

$$\Rightarrow \quad \frac{\tan^{-1} y - x}{1 + y^2}dy = dx \quad ...(i)$$

Putting $\tan^{-1} y = t$ and $\frac{1}{1+y^2} = \frac{dt}{dy}$ in (i), we get

$(t - x)\,dt = dx$

$$\Rightarrow \quad \frac{dx}{dt} = t - x$$

$$\Rightarrow \frac{dx}{dt} + x = t \quad ...(ii)$$

Here, I.F. $= e^{\int 1.dt} = e^t$

Hence, the solution of the differential equation (ii) is given by

$$x(\text{I.F.}) = \int(\text{I.F.})\, t\, dt$$

$xe^t = \int e^t t\,dt + C$, where C is arbitrary constant

$$\Rightarrow \quad xe^t = te^t - \int e^t dt + C$$

$$\Rightarrow \quad xe^t = te^t - e^t + C$$

$$\Rightarrow xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C \quad ...(iii)$$

$[\because t = \tan^{-1} y]$

It is given that $x = 1$ when $y = 0$

So, $e^0 = e^0(0 - 1) + C$

$\Rightarrow$ $C = 2$

Putting $C = 2$ in (iii) we get

$$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1} y - 1) + 2$$

$$\therefore \quad x = \tan^{-1}y + 2e^{\tan y} - 1$$

is the particular solution the given differential equation. **Ans.**

24. Find the distance of the point P(3, 4, 4) from the point, where the line joining the points A(3, – 4, – 5) and B(2, –3, 1) intersects the plane $2x + y + z = 7$. **[6]**

Solution : Equation of the line joining the points A(3, – 4, – 5) and B(2, – 3, 1) is

$$\Rightarrow \quad \frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = k.$$

Coordinates of any point on the line are

$Q\,(3 - k, k - 4, 6k - 5)$

The given line intersects the plane $2x + y + z = 7$.

So, $2(3 - k) + k - 4 + 6k - 5 = 7$

$\Rightarrow$ $6 - 2k + k - 4 + 6k - 5 = 7$

$\Rightarrow$ $5k = 10$

$\Rightarrow$ $k = 2$

$\therefore$ Coordinates of a point where line Intersect plane are $(3 - k, k - 4, 6k - 5) = (3 - 2, 2 - 4, 12 - 5) = (1, - 2, 7)$

Now, the distance between the point P(3, 4, 4) and A(1, – 2, 7) is given by

$$PQ = \sqrt{(3-1)^2 + (4+2)^2 + (4-7)^2}$$

$$\Rightarrow \quad PQ = \sqrt{(2)^2 + (6)^2 + (-3)^2}$$

$$\Rightarrow \quad PQ = \sqrt{4 + 36 + 9} = \sqrt{49}$$

$\therefore$ PQ = 7 units. **Ans.**

25. A company manufactures three kinds of calculators : A, B and C in its two factories I and II. The company has got an order for

manufacturing at least 6400 calculators of kind A, 4000 of kind B and 4800 of kind C. The daily output of factory I is of 50 calculators of kind A, 50 calculators of kind B, and 30 calculators of kind C. The daily output of factory II is of 40 calculators of kind A, 20 of kind B and 40 of kind C. The cost per day to run factory I is ₹12,000 and of factory II is ₹15,000. How many days do the two factories have to be in operation to produce the order with the minimum cost? Formulate this problem as an LPP and solve it graphically. **[6]**

Solution : Let the factories I and II work for x and y number of days respectively.

Thus, the given linear programming problem is Minimize Z = ₹ $(12000x + 15000y)$

Subject to the constraints

$$50x + 40y \geq 6400$$
$$50x + 20y \geq 4000$$
$$30x + 40y \geq 4800$$
$$x \geq 0$$
and $$y \geq 0$$
i.e. $$5x + 4y \geq 640$$
$$5x + 2y \geq 400$$
$$3x + 4y \geq 480$$
$$x \geq 0, y \geq 0.$$

To solve this L. P. P,

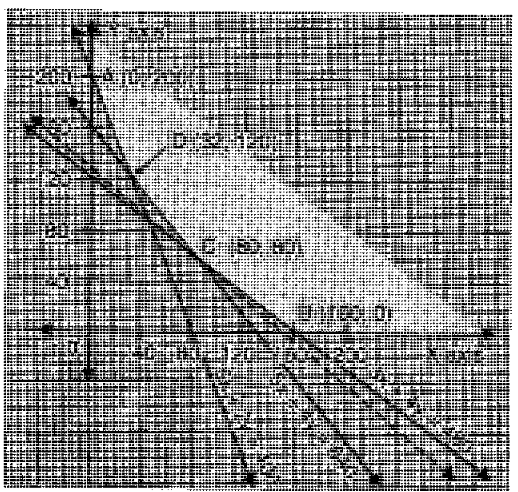
Let us consider the equations

$L_1 : 5x + 4y = 640$ ...(i)

$L_2 : 5x + 2y = 400$ ...(ii)

$L_3 : 3x + 4y = 480$ ...(iii)

The point of intersection of L_1 and L_2 is D(32, 120) and that of L_1 and L_3 is C (80,60)



The shaded region is the solution region of the given L. P. P.

Corner Points	Values of the objective function Z=12000x + 15000y
A (0, 200)	$12000 \times 0 + 15000 \times 200 = 30,00,000$
B (160,0)	$12000 \times 160 + 15000 \times 0 = 19,20,000$
C (80, 60)	$12000 \times 80 + 15000 \times 60 = 18,60,000$
D (32,120)	$12000 \times 32 + 15000 \times 120 = 21,84,000$

Out of these values of Z, the minimum value of Z is 18,60,000 at $x = 80$ and $y = 60$.

Since the feasible region is unbounded so we draw the graph of inequality

$$12000x + 15000y < 1860000$$
i.e., $$4x + 5y \leq 620$$

L : $4x + 5y = 620$

x	0	155
y	124	0

We observe that open half one represented by L have no point common with feasible region.

$$Z = 12000 \times 80 + 15000 \times 60$$
$$= ₹\ 18,60,000.$$

Hence, the factories I and II work for 80 and 60 number of days respectively. **Ans.**

26. In a factory which manufactures bolts, machines A, B and C manufacture respectively 30%, 50% and 20% of the bolts. Of their outputs 3, 4 and 1 percent respectively are defective bolts. A bolt is drawn at random from the product and is found to be defective. Find the probability that this is not manufactured by machine B. **[6]**

Solution : Let E_1, E_2, E_3 and A be the events defined as below

E_1 = the bolt is manufactured by machine A.

E_2 = the bolt is manufactured by machine B.

E_3 = the bolt is manufactured by machine C.

A = the bolt is defective.

then, $P(E_1)$ = Probability that the bolt drawn is manufactured by machine A = $\frac{30}{100}$

$P(E_2)$ = Probability that the bolt drawn is manufactured by machine B = $\frac{50}{100}$

$P(E_3)$ = Probability that the bolt drawn is manufactured by machine C = $\frac{20}{100}$

$P(A/E_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A.

$$P(A/E_1) = \frac{3}{100}$$

Similarly, we have, $$P(A/E_2) = \frac{4}{100}$$

and $$P(A/E_3) = \frac{1}{100}$$

Now, using Bayes' theorem

$P(E_2/A)$ = Probability that the bolt is manufactured by machine B given that the bolt drawn is defective.

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{50}{100}\times\frac{4}{100}}{\frac{30}{100}\times\frac{3}{100}+\frac{50}{100}\times\frac{4}{100}+\frac{20}{100}\times\frac{1}{100}}$$

$$= \frac{200}{90+200+20} = \frac{200}{310} = \frac{20}{31}.$$

Hence, the probability that this is not manufactured by Machine B = $1 - P(E_2/A)$

$$= 1-\frac{20}{31} = \frac{11}{31}$$ **Ans.**

●●

All questions are same in Outside Delhi Set II and Set III

Mathematics 2015 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. If $\vec{a} = 7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$. **[1]**

Solution : Given,

$$\vec{a} = 7\hat{i}+\hat{j}-4\hat{k}$$

$$\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$$

∴ The projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$= \frac{(7\times2)+(1\times6)+(-4\times3)}{\sqrt{(2)^2+(6)^2+(3)^2}} = \frac{8}{7}$$

Ans.

2. Find λ, if the vectors $\vec{a} = \hat{i}+3\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}-\hat{k}$ and $\vec{c} = \lambda\hat{j}+3\hat{k}$ are coplanar. **[1]**

Solution : Since, the given vectors are coplanar.

$$[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

Here, $$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & \lambda & 3 \end{vmatrix} = 0$$

Expanding along R_3, we get

$$0.(-3+1) - \lambda(-1-2) + 3(-1-6) = 0$$

$$\Rightarrow \quad 3\lambda = 21$$

$$\therefore \quad \lambda = 7$$ **Ans.**

3. If a line makes angles 90°, 60° and θ with x, y and z–axis respectively, where θ is acute, then find θ. **[1]**

Solution : Given,

$$\alpha = 90^\circ$$
$$\beta = 60^\circ$$
$$\lambda = \theta$$

Let l, m, n be the direction cosines of the given vector.

Then, $l = \cos\alpha$,

$m = \cos\beta$

and $n = \cos\lambda$

Now, $$l^2+m^2+n^2 = 1$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\lambda = 1$$

$$\Rightarrow \cos^2(90^\circ)+\cos^2(60^\circ)+\cos^2\theta = 1$$

$$\Rightarrow 0^2+\left(\frac{1}{2}\right)^2+\cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1-\frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \quad (\because \theta \text{ is acute})$$

$$\therefore \theta = 30^\circ$$ **Ans.**

4. Write the element a_{23} of a 3 × 3 matrix $A = (a_{ij})$ whose elements a_{ij} are given be $a_{ij} = \dfrac{|i-j|}{2}$. **[1]**

Solution : Given,

$$a_{ij} = \frac{|i-j|}{2}$$

$\therefore \quad a_{23} = \frac{|2-3|}{2} = \frac{|-1|}{2} = \frac{1}{2}$ **Ans.**

5. Find the differential equation representing the family of curves $v = \frac{A}{r} + B$, where A and B are arbitrary constants. [1]

Solution : We have,

$$v = \frac{A}{r} + B \quad ...(i)$$

Since, the given equation contains two arbitrary constants, we shall differentiate it two times. Now, differentiating (i) w.r.t. r, we get

$$\frac{dv}{dr} = -\frac{A}{r^2} + 0$$

$$\Rightarrow \quad r^2 \frac{dv}{dr} = -A \quad ...(ii)$$

Again, differentiating (ii) w.r.t. r, we get

$$r^2 \times \frac{d^2v}{dr^2} + 2r \times \frac{dv}{dr} = 0$$

$$\Rightarrow \quad r\frac{d^2v}{dr^2} + 2\frac{dv}{dr} = 0$$

This is the required differential equation representing the family of the given curve. **Ans.**

6. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1.$ [1]

Solution : We have,

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}},$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \frac{1}{\sqrt{x}}$

and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \quad \text{I. F.} = e^{\int P dx}$$

$$= e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$$ **Ans.**

SECTION — B

7. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = 0$. [4]

Solution : We have,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad A^2 = A.A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\Rightarrow \quad -5A = \begin{bmatrix} (-5).2 & (-5).0 & (-5).1 \\ (-5).2 & (-5).1 & (-5).3 \\ (-5).1 & (-5)(-1) & (-5).0 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix}$$

$$\Rightarrow \quad 4I_3 = 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\therefore \quad A^2 - 5A + 4I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

Now, $A^2 - 5A + 4I + X = 0$

$$\Rightarrow \quad X = -(A^2 - 5A + 4I)$$

$$\Rightarrow \quad X = (-1)\begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$$

Ans.

OR

If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A')^{-1}$.

Solution : Given,

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$$

Now, $$A' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$|A'| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= 1(-1-8) - 0(-2-6) - 2(-8+3)$$
$$= -9 + 10 = 1 \neq 0$$

So, A′ is invertible.

$$A'_{11} = -9, A'_{12} = 8, A'_{13} = -5$$
$$A'_{21} = -8, A'_{22} = 7, A'_{23} = -4$$
$$A'_{31} = -2, A'_{32} = 2, A'_{33} = -1$$

$$\text{adj } A' = \begin{bmatrix} A'_{11} & A'_{21} & A'_{31} \\ A'_{12} & A'_{22} & A'_{32} \\ A'_{13} & A'_{23} & A'_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$\therefore \quad (A')^{-1} = \frac{1}{|A'|} \text{adj } A'$$

$$= \text{adj } A' \quad [\because |A'| = 1]$$

$$\Rightarrow \quad (A')^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$ **Ans.**

8. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$**, using properties of determinants find the value of** $f(2x) - f(x)$**. [4]**

Solution : Given, $$f(x) = \begin{bmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{bmatrix}$$

Taking a common from C_1

$$\Rightarrow \quad f(x) = a\begin{bmatrix} 1 & -1 & 0 \\ x & a & -1 \\ x^2 & ax & a \end{bmatrix}$$

Applying $C_2 \to C_2 + C_1$, we get

$$f(x) = a\begin{bmatrix} 1 & 0 & 0 \\ x & x+a & -1 \\ x^2 & x^2+ax & a \end{bmatrix}$$

On expanding along R_1,

$$f(x) = a(a^2 + ax + ax + x^2)$$
$$\Rightarrow \quad f(x) = a(a^2 + 2ax + x^2)$$

Also, $$f(2x) = \begin{vmatrix} a & -1 & 0 \\ 2ax & a & -1 \\ 4ax^2 & 2ax & a \end{vmatrix}$$

$$\Rightarrow \quad f(2x) = a\begin{vmatrix} 1 & -1 & 0 \\ 2x & a & -1 \\ 4x^2 & 2ax & a \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, we get

$$f(2x) = a\begin{vmatrix} 1 & 0 & 0 \\ 2x & 2x+a & -1 \\ 4x^2 & 4x^2+2ax & a \end{vmatrix}$$

On Expanding along R_1

$$\Rightarrow \quad f(2x) = a\{a(2x+a) + 4x^2 + 2ax\}$$
$$\Rightarrow \quad f(2x) = a\{4x^2 + a^2 + 4ax\}$$
$$\therefore \quad f(2x) - f(x) = a(4x^2 + a^2 + 4ax - a^2 - 2ax - x^2)$$
$$\Rightarrow \quad = a(3x^2 + 2ax)$$
$$\Rightarrow \quad = ax(3x + 2a).$$ **Ans.**

9. Find : $\int \frac{dx}{\sin x + \sin 2x}$. **[4]**

Solution : Let $$I = \int \frac{1}{\sin x + \sin 2x} dx$$

$$\Rightarrow \quad I = \int \frac{1}{\sin x + 2\sin x \cos x} dx$$

$$\Rightarrow \quad I = \int \frac{1}{\sin x(1 + 2\cos x)} dx$$

$$\Rightarrow \quad I = \int \frac{\sin x}{\sin^2 x(1 + 2\cos x)} dx$$

$$= \int \frac{\sin x\, dx}{(1 - \cos^2 x)(1 + 2\cos x)}$$

Putting $\cos x = t, -\sin x\, dx = dt$

$$\Rightarrow \quad \sin x\, dx = -dt$$

$$I = \int \frac{-dt}{(1-t^2)(1+2t)}$$

$$= \int \frac{-1}{(1-t)(1+t)(1+2t)} dt$$

Let $\frac{-1}{(1+t)(1-t)(1+2t)} = \frac{A}{(1+t)} + \frac{B}{(1-t)} + \frac{C}{(1+2t)}$ ...(i)

$-1 = A(1-t)(1+2t) + B(1+t)(1+2t) + C(1+t)(1-t)$

Putting $1-t = 0$

or $t = 1$ in (i), we get

$-1 = 6B$

$\Rightarrow B = \frac{-1}{6}$

Putting $1+t = 0$

or $t = -1$ in (i), we get

$-1 = -2A$

$\Rightarrow A = \frac{1}{2}$

Putting $1+2t = 0$

or $t = -\frac{1}{2}$ in (i), we get

$-1 = \frac{3}{4}C$

$\Rightarrow C = -\frac{4}{3}$

$$\frac{-1}{(1+t)(1-t)(1+2t)} = \frac{1}{2(1+t)} - \frac{1}{6(1-t)} - \frac{4}{3(1+2t)}$$

$$\Rightarrow I = \int \frac{-dt}{(1+t)(1-t)(1+2t)}$$

$$\Rightarrow I = \frac{1}{2}\int \frac{1}{1+t}dt - \frac{1}{6}\int \frac{1}{1-t}dt - \frac{4}{3}\int \frac{1}{1+2t}dt$$

$$\Rightarrow I = \frac{1}{2}\log|1+t| + \frac{1}{6}\log|1-t| - \frac{4}{3\times 2}\log|1+2t| + C$$

$$\Rightarrow I = \frac{1}{2}\log|1+\cos x| + \frac{1}{6}\log|1-\cos x| - \frac{2}{3}\log|1+2\cos x| + C$$

Ans.

OR

Integrate the following w.r.t. x : $\frac{x^2-3x+1}{\sqrt{1-x^2}}$. **[1]**

Solution : Given,

$$\frac{x^2-3x+1-1+1}{\sqrt{1-x^2}} = -\left[\frac{-x^2+3x-1+1-1}{\sqrt{1-x^2}}\right]$$

$$= -\left[\frac{1-x^2+3x-2}{\sqrt{1-x^2}}\right]$$

$$= -\frac{1-x^2}{\sqrt{1-x^2}} - \frac{3x-2}{\sqrt{1-x^2}}$$

$$= -\sqrt{1-x^2} - \frac{3x-2}{\sqrt{1-x^2}}$$

$$\Rightarrow \int \frac{x^2-3x+1}{\sqrt{1-x^2}}dx = \int\left(-\sqrt{1-x^2} - \frac{3x-2}{\sqrt{1-x^2}}\right)dx$$

$$= -\int\sqrt{1-x^2}dx - \int\frac{3x-2}{\sqrt{1-x^2}}dx$$

$$= -\int\sqrt{1-x^2}dx - 3\int\frac{x}{\sqrt{1-x^2}}dx + 2\int\frac{1}{\sqrt{1-x^2}}dx$$

$$= -\int\sqrt{1-x^2}dx + \frac{3}{2}\int\frac{1}{\sqrt{t}}dt + 2\int\frac{1}{\sqrt{1-x^2}}dx$$

(Putting $1-x^2 = t \Rightarrow -2x\,dx = dt$)

$$= \frac{-x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x + 3\sqrt{1-x^2} + 2\sin^{-1}x + C$$

$$= \frac{-x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1}x + 3\sqrt{1-x^2} + C$$ **Ans.**

10. Evaluate : $\int_{-\pi}^{\pi}(\cos ax - \sin bx)^2 dx$. **[4]**

Solution :

Let $I = \int_{-\pi}^{\pi}(\cos ax - \sin bx)^2 dx$

$I = \int_{-\pi}^{\pi}(\cos^2 ax + \sin^2 bx - 2\cos ax \sin bx)dx$

$I = \int_{-\pi}^{\pi}(\cos^2 ax\,dx + \int_{-\pi}^{\pi}\sin^2 bx\,dx - 2\int_{-\pi}^{\pi}\cos ax \sin bx\,dx$

$\Rightarrow I = 2\int_0^{\pi}\cos^2 ax\,dx + 2\int_0^{\pi}\sin^2 bx\,dx - 0$

[Since $\cos^2 ax$ and $\sin^2 bx$ are even functions and $\cos ax \sin bx$ is an odd function]

$$\Rightarrow I = 2\int_0^{\pi}\left(\frac{1+\cos 2ax}{2}\right)dx + 2\int_0^{\pi}\left(\frac{1-\cos 2bx}{2}\right)dx$$

$$\Rightarrow I = \int_0^{\pi}(1+\cos 2ax)dx + \int_0^{\pi}(1-\cos 2bx)\,dx$$

$$\Rightarrow I = \int_0^{\pi}(1+\cos 2ax + 1 - \cos 2bx)dx$$

$$\Rightarrow I = \int_0^{\pi}(2+\cos 2ax - \cos 2bx)dx$$

$$\Rightarrow I = 2[x]_0^{\pi} + \left[\frac{\sin 2ax}{2a}\right]_0^{\pi} - \left[\frac{\sin 2bx}{2b}\right]_0^{\pi}$$

$$\Rightarrow I = 2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b}$$

$\Rightarrow I = 2\pi$ if $a, b \in Z$ **Ans.**

11. A bag 'A' contains 4 black and 6 red balls and bag 'B' contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black. [4]

Solution : Consider the following events :

E_1 = Getting 1 or 2 on die.

E_2 = Getting 3, 4, 5 or 6 on die.

E = One of the ball drawn is red and another is black.

$$P(E_1) = \frac{2}{6} = \frac{1}{3}$$

and $$P(E_2) = \frac{4}{6} = \frac{2}{3}$$

$P(E/E_1)$ = Probability of drawing a red and a black ball when bag A has been chosen.

$$= P(RB) + P(BR)$$

$$P(E/E_1) = \frac{6}{10}\times\frac{4}{9}+\frac{4}{10}\times\frac{6}{9} = \frac{48}{90} = \frac{8}{15}$$

$P(E/E_2)$ = Probability of drawing a red and a black ball when bag B has been chosen.

$$= \frac{3}{10}\times\frac{7}{9}+\frac{7}{10}\times\frac{3}{9} = \frac{42}{90} = \frac{7}{15}$$

Using the law of total probability, we have

$$P(E) = P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2)$$

$$= \frac{1}{3}\times\frac{8}{15}+\frac{2}{3}\times\frac{7}{15}$$

$$= \frac{22}{45}$$ **Ans.**

OR

An unbiased coin is tossed 4 times. Find the mean and variance of the number of heads obtained.

Solution : Let X denote the number of heads in the four tosses of the coin, then X is a random variable that can have values 0, 1, 2, 3, 4.

$P(X = 0)$ = Probability of getting no head (TTTT)

$$= \frac{1}{16}$$

$P(X = 1)$ = Probability of getting one head (HTTT, THTT, TTHT, TTTH)

$$= 4\times\frac{1}{16} = \frac{1}{4}$$

$P(X = 2)$ = Probability of getting two head (HHTT, HTHT, HTTH, THHT, THTH, TTHH)

$$= 6\times\frac{1}{16} = \frac{3}{8}$$

$P(X = 3)$ = Probability of getting three head (HHHT, HHTH, HTHH, THHH)

$$= 4\times\frac{1}{16} = \frac{1}{4}$$

$P(X = 4)$ = Probability of getting four head (HHHH)

$$= \frac{1}{16}$$

Thus, the probability distribution of random variable X is given by

X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

x_i	$P_i = P(X = x_i)$	$P_i x_i$	$P_i x_i^2$
0	$\frac{1}{16}$	0	0
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{2}$
3	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{9}{4}$
4	$\frac{1}{16}$	$\frac{1}{4}$	1
		$\Sigma P_i x_i = 2$	$\Sigma P_i x_i^2 = 5$

$\therefore$ Mean $= \bar{X} = \Sigma P_i x_i = 2$

and $\text{Var}(x) = \Sigma P_i x_i^2 - (\Sigma P_i x_i)^2 = 5 - 4 = 1$

Hence, Mean = 2 and Variance = 1. **Ans.**

12. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find $(\vec{r}\times\hat{i}).(\vec{r}\times\hat{j}) + xy$. [4]

Solution : Given,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$$

Now, $(\vec{r}\times\hat{i}).(\vec{r}\times\hat{j}) + xy$

$$= [(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}].[(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j}] + xy$$

$$[\because \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}]$$

$$= [(x\hat{i}\times\hat{i}+y\hat{j}\times\hat{i}+z\hat{k}\times\hat{i})].[(x\hat{i}\times\hat{j}+y\hat{j}\times\hat{j}$$

$$+ z\hat{k} \times \hat{j})] + xy$$

$$\left[\begin{array}{l} \because \hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i}, \\ \hat{k} \times \hat{i} = \hat{j},\ \hat{i} \times \hat{i} = 0 \\ \hat{j} \times \hat{j} = 0,\ \hat{k} \times \hat{k} = 0 \end{array}\right]$$

$$= (0 - y\hat{k} + z\hat{j}).(x\hat{k} + 0 - z\hat{i}) + xy$$

$$= (0z - xy + 0z) + xy$$

$$= -xy + xy = 0.$$

Ans.

13. Find the distance between the point (– 1, – 5, – 10) and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$. **[4]**

Solution : Let $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = k$

$$\Rightarrow \quad x = 3k + 2,$$
$$y = 4k - 1,$$
$$z = 12k + 2$$

Coordinates of any point on the line are

$(3k + 2, 4k - 1, 12k + 2)$.

The point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ will also be in the form $(3k + 2, 4k - 1, 12k + 2)$ and it will satisfy the equation of plane.

Now, putting $x = 3k + 2,$

$y = 4k - 1$

and $z = 12k + 2$ in $x - y + z = 5,$

we get

$$3k + 2 - (4k - 1) + 12k + 2 = 5$$
$$\Rightarrow \quad 11k + 5 = 5$$
$$\Rightarrow \quad 11k = 0$$
$$\Rightarrow \quad k = 0$$
$$\therefore \quad x = 2,$$
$$y = -1,$$
$$z = 2$$

Hence, the point of intersection of $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ is $(2, -1, 2)$.

$\therefore$ Distance between the point $(-1, -5, -10)$ and $(2, -1, 2)$

Using distance formula

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$
$$= \sqrt{3^2 + 4^2 + 12^2}$$
$$= \sqrt{169} = 13 \text{ units}$$

Hence, the distance between the point (–1, –5, – 10) and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ is 13 units.

Ans.

14. If $\sin[\cot^{-1}(x + 1)] = \cos(\tan^{-1} x)$, then find x. **[4]**

Solution : Given, $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1} x)$

$$\sin\left\{\sin^{-1}\frac{1}{\sqrt{1+(x+1)^2}}\right\} = \cos\left\{\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right\}$$

$$\left[\because \cot^{-1} x = \sin^{-1}\frac{1}{\sqrt{1+x^2}} \text{ and } \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+(x+1)^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^2+1+2x}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \sqrt{1+x^2} = \sqrt{x^2+2x+2}$$

On squaring both sides, we get

$$1 + x^2 = x^2 + 2x + 2$$
$$\Rightarrow \quad 2x + 2 = 1$$
$$\Rightarrow \quad x = \frac{-1}{2}$$

Ans.

OR

If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$, then find x.

Solution : Given, $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$

$$\Rightarrow \quad (\tan^{-1} x + \cot^{-1} x)^2 - 2\tan^{-1} x \cot^{-1} x = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1} x\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad \frac{\pi^2}{4} - \pi\tan^{-1} x + 2(\tan^{-1} x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad 2(\tan^{-1} x)^2 - \pi\tan^{-1} x + \frac{\pi^2}{4} - \frac{5\pi^2}{8} = 0$$

$$\Rightarrow \quad 2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \frac{3\pi^2}{8} = 0$$

Solving the quadratic equation, we get

$$\tan^{-1}x = \frac{\pi \pm \sqrt{\pi^2 + 4\times 2\times \frac{3\pi^2}{8}}}{2\times 2}$$

$$\Rightarrow \quad \tan^{-1}x = \frac{\pi \pm 2\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}x = \frac{3\pi}{4} \text{ or } \tan^{-1}x = \frac{-\pi}{4}$$

$$\Rightarrow \quad x = \tan\frac{3\pi}{4} \text{ or } x = \tan\left(\frac{-\pi}{4}\right)$$

$$\Rightarrow \quad x = -1.$$ **Ans.**

15. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, $x^2 \le 1$, then find $\frac{dy}{dx}$. **[4]**

Solution : Given,

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$$

Putting $x^2 = \cos 2\theta$ we have

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

Dividing the numerator and denominator by $\cos\theta$, we get

$$y = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}.\tan\theta}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{4}+\theta$$

$$\therefore \quad y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \quad (\because x^2 = \cos 2\theta) \quad ...(i)$$

Now, differentiating (i), w.r.t. x, we get

$$\frac{dy}{dx} = 0+\frac{1}{2}\times\left(\frac{-1}{\sqrt{1-(x^2)^2}}\right)\times 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^4}}$$ **Ans.**

16. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$. **[4]**

Solution : We have,

$$x = a\cos\theta + b\sin\theta \quad ...(i)$$

$$y = a\sin\theta - b\cos\theta \quad ...(ii)$$

On squaring and adding (i) and (ii), we get

$$x^2 + y^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$$

$$= a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta + a^2\sin^2\theta + b^2\cos^2\theta - 2ab\cos\theta\sin\theta$$

$$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow x^2 + y^2 = a^2 + b^2 \quad ...(iii)$$

Now, differentiating both sides of equation (iii) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \quad ...(iv)$$

Again, differentiating both sides of (iv) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left(\frac{y\times 1 - x\times\frac{dy}{dx}}{y^2}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{y - x\left(-\frac{x}{y}\right)}{y^2}\right] \quad \text{[from (iv)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{y^2+x^2}{y^3}\right] \quad ...(v)$$

Now, we have

$$\text{L. H. S.} = y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y$$

$$= y^2\left(\frac{-y^2-x^2}{y^3}\right) - x\left(-\frac{x}{y}\right) + y$$

$$= \frac{-y^2-x^2}{y} + \frac{x^2}{y} + y$$

$$= \frac{-y^2 - x^2 + x^2 + y^2}{y}$$

$= 0 =$ R. H. S. **Hence Proved.**

17. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm ? [4]

Solution : We know that, Area of an equilateral triangle,

$$A = \frac{\sqrt{3}}{4}a^2,$$

where a = side of an equilateral triangle.

Given $\frac{da}{dt} = 2$ cm/s

Now, $$\frac{dA}{dt} = \frac{d}{dt}\left(\frac{\sqrt{3}}{4}a^2\right)$$

$$= \frac{\sqrt{3}}{4} \times 2 \times a \times \frac{da}{dt}$$

$$= \left(\frac{\sqrt{3}}{2}a.\frac{da}{dt}\right)$$

$$= \frac{\sqrt{3}a}{2}.2 = \sqrt{3}a \text{ cm}^2/\text{s}$$

when the side of the triangle is 20 cm.

$$\therefore \left[\frac{dA}{dt}\right]_{a=20} = 20\sqrt{3} \text{ cm}^2/\text{s}$$

Hence, the area is increasing at the rate of $20\sqrt{3}$ cm^2/s when the side of the triangle is 20 cm. **Ans.**

18. Find $\int (x+3)\sqrt{3-4x-x^2}\,dx$. [4]

Solution :

Let $I = \int (x+3)\sqrt{3-4x-x^2}\,dx$

Let $x + 3 = \lambda \frac{d}{dx}(3-4x-x^2) + \mu$

$\Rightarrow x + 3 = \lambda(-4-2x) + \mu$

$\Rightarrow x + 3 = (-2\lambda)x - 4\lambda + \mu$

Equating the coefficients of like terms, we get

$-2\lambda = 1$

$\Rightarrow \lambda = \frac{-1}{2}$

and $-4\lambda + \mu = 3$

$\Rightarrow -4\left(\frac{-1}{2}\right) + \mu = 3$

$\Rightarrow \mu = 3 - 2 = 1$

$$\therefore I = \int \left[\frac{1}{2}(4+2x)+1\right]\sqrt{3-4x-x^2}\,dx$$

$$= \int \frac{1}{2}(4+2x)\sqrt{3-4x-x^2}\,dx + \int \sqrt{3-4x-x^2}\,dx$$

$$= \frac{1}{2}I_1 + I_2 \quad \text{...(i)}$$

Now, $I_1 = \int (4+2x)\sqrt{3-4x-x^2}\,dx$

Let $3 - 4x - x^2 = t$

$\Rightarrow -(4+2x)dx = dt$

$$\therefore I_1 = -\int \sqrt{t}\,dt$$

$$= -\int t^{\frac{1}{2}} dt$$

$$= -\frac{2}{3}t^{\frac{3}{2}} + C_1$$

$$= -\frac{2}{3}(3-4x-x^2)^{\frac{3}{2}} + C_1 \quad \text{...(ii)}$$

$(\because t = 3 - 4x - x^2)$

and $I_2 = \int \sqrt{3-4x-x^2}\,dx$

$$= \int \sqrt{3+4-(x^2+4x+4)}\,dx$$

$$= \int \sqrt{7-(x+2)^2}\,dx$$

$$= \int \sqrt{(\sqrt{7})^2-(x+2)^2}\,dx$$

$$= \frac{1}{2}(x+2)\sqrt{(\sqrt{7})^2-(x+2)^2} + \frac{(\sqrt{7})^2}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C_2$$

$$= \frac{1}{2}(x+2)\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C_2 \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$I = -\frac{2}{3}(3-4x-x^2)^{\frac{3}{2}} + \frac{1}{2}(x+2)\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C,$$

where $C = C_1 + C_2$ **Ans.**

19. Three schools A, B and C organised a mela for collecting funds for helping the rehabilitation of flood victims. They sold handmade fans, mats and plates from recycled material at a cost of ₹ 25, ₹ 100 and ₹ 50 each. The number of articles sold are given below :

Article \ School	A	B	C
Hand-fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the fund collected by each school separately by selling the above articles. Also find the total funds collected for the purpose. Write one value generated by the above situation. [4]

Solution : The number of articles sold by each school can be represented by the 3 × 3 matrix

$$X = \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$$

and the cost of each article can be represented by the 1 × 3 matrix

$$Y = [2510050]$$

∴ Funds collected by each school separately is given by the matrix multiplication.

$$YX = [25 \ \ 100 \ \ 50] \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$$

$$YX = [7000 \ \ 6125 \ \ 7875]$$

Hence, the funds collected by schools A, B and C are ₹ 7,000, ₹ 6,125 and ₹ 7,875 respectively.

The total funds collected for flood victims

$$= ₹(7{,}000 + 6{,}125 + 7{,}875)$$
$$= ₹\ 21{,}000$$

The above situation exhibits the helping nature of students. **Ans.**

SECTION – C

20. Let N denote the set of all natural numbers and R be the relation on N × N defined by $(a, b)\ R\ (c, d)$ if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation. [6]

Solution : We know that relation R will be an equivalence relation, if we prove it as a reflexive, symmetric and transitive relation.

(i) Reflexivity :

Let (a, b), be an arbitrary element of N × N

then, $(a, b) \in N \times N$

$a, b \in N$

$\Rightarrow \quad ab(b + a) = ba(a + b)$

$\Rightarrow \quad (a, b)\ R\ (a, b)$

$\therefore \quad (a, b)\ R\ (a, b)\ \forall\ (a, b) \in N \times N.$

(ii) Symmetry :

Let $(a, b), (c, d)$ be an arbitrary element of N × N such that $(a, b)\ R\ (c, d)$

$\Rightarrow \quad ad(b + c) = bc(a + d)$

$\Rightarrow \quad cb(d + a) = da(c + b)$

$\Rightarrow \quad (c, d)\ R\ (a, b)$

$\therefore (a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)\ \forall\ (a, b), (c, d) \in N \times N$

So, R is symmetric on N × N.

(iii) Transitivity :

Let $(a, b), (c, d), (e, f)$ be an arbitrary element of N × N such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$, then $(a, b)\ R\ (c, d)$

$\Rightarrow \quad ad(b + c) = bc(a + d)$

$\Rightarrow \quad \frac{b+c}{bc} = \frac{a+d}{ad}$

$\Rightarrow \quad \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d}$ ...(i)

Also, $(c, d)\ R\ (e, f)$

$\Rightarrow \quad cf(d + e) = de(c + f)$

$\Rightarrow \quad \frac{d+e}{de} = \frac{c+f}{cf}$

$\Rightarrow \quad \frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f}$ ...(ii)

Adding (i) and (ii), we get

$$\left(\frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{d} + \frac{1}{e}\right) = \left(\frac{1}{a} + \frac{1}{d}\right) + \left(\frac{1}{c} + \frac{1}{f}\right)$$

$\Rightarrow \quad \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f}$

$\Rightarrow \quad \frac{b+e}{be} = \frac{a+f}{af}$

$\Rightarrow \quad af(b + e) = be(a + f)$

$\Rightarrow \quad (a, b)\ R\ (e, f)$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$. $\forall\ (a, b)\ (c, d)\ (e, f) \in N \times N$

So, R is transitive on N × N.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on N × N.

Hence Proved.

21. Using integration find the area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$. [6]

Solution : The equation of the given circle is $x^2 + y^2 = 4$. The equation of the normal to the circle at $(1, \sqrt{3})$ is same as the line joining the points $(1, \sqrt{3})$ and $(0, 0)$ which is given by

$$y - 0 = \frac{\sqrt{3} - 0}{1 - 0}(x - 0)$$

$$\left[\because y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\right]$$

$\Rightarrow \quad y = \sqrt{3}x \quad$ ...(i)

So, the slope of the normal is $\sqrt{3}$.

We know that, slope of normal × slope of tangent $= -1$.

$\therefore$ the slope of tangent $= \frac{-1}{\sqrt{3}}$

Now, the equation of the tangent to the circle at $(1, \sqrt{3})$ is given by :

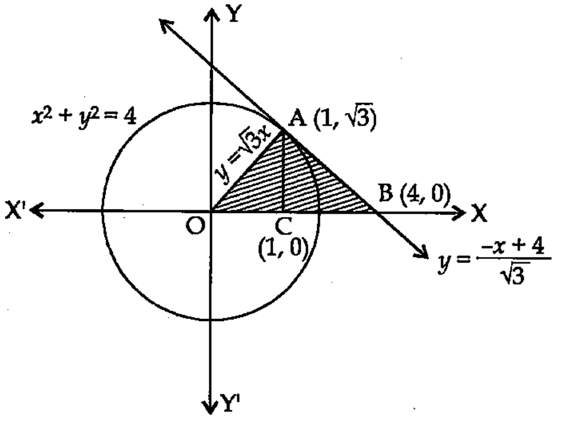
$$y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x-1) \quad [\because y - y_1 = m(x - x_1)]$$

$$\Rightarrow \sqrt{3}y - 3 = -x + 1$$

$$\Rightarrow y = \frac{-x+4}{\sqrt{3}} \quad \text{...(ii)}$$

Putting $y = 0$ in (ii), we get $x = 4$.

Thus, AOB is triangle formed by the tangent, normal and the positive x-axis.



Now, Area of Δ AOB = Area of Δ AOC + Area of Δ ACB.

$$= \int_0^1 y\,dx + \int_1^4 y\,dx$$

$$= \int_0^1 \sqrt{3}\,x\,dx + \int_1^4 \left(\frac{-x+4}{\sqrt{3}}\right)dx$$

$$= \sqrt{3}\int_0^1 x\,dx - \frac{1}{\sqrt{3}}\int_1^4 x\,dx + \frac{4}{\sqrt{3}}\int_1^4 1.dx$$

$$= \sqrt{3}\left[\frac{x^2}{2}\right]_0^1 - \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_1^4 + \frac{4}{\sqrt{3}}[x]_1^4$$

$$= \sqrt{3}\left(\frac{1}{2} - 0\right) - \frac{1}{\sqrt{3}}\left(\frac{16}{2} - \frac{1}{2}\right) + \frac{4}{\sqrt{3}}(4-1)$$

$$= \frac{\sqrt{3}}{2} - \frac{15}{2\sqrt{3}} + \frac{12}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2} + \frac{9}{2\sqrt{3}} = \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}$$

$$= 2\sqrt{3} \text{ sq. units.}$$

Hence, the area of the tringle so formed is $2\sqrt{3}$ square units. **Ans.**

OR

Evaluate $\int_1^3 (e^{2-3x} + x^2 + 1)\,dx$ as a limit of a sum.

Solution : We have, $\int_a^b f(x)dx$

$$= \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + ... + f(a + (n-1)h)]$$

where $h = \frac{b-a}{n}$

Here, $a = 1, b = 3$

and $f(x) = e^{2-3x} + x^2 + 1$

$\therefore \quad h = \frac{2}{n}$

$\Rightarrow \quad hn = 2$

Now, $f(a) = f(1)$

$$= e^{2-3\times 1} + 1^2 + 1$$

$$f(a+h) = f(1+h) = e^{2-3(1+h)} + (1+h)^2 + 1$$

$$f(a+2h) = f(1+2h) = e^{2-3(1+2h)} + (1+2h)^2 + 1$$

$$f(a+(n-1)h) = f(1+(n-1)h) = e^{2-3[1+(n-1)h]} + [1+(n-1)h]^2 + 1$$

Adding these equations, we get

$$f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)$$
$$= [e^{2-3\times 1} + 1^2 + 1] + [e^{2-3\times(1+h)} + (1+h)^2 + 1] + [e^{2-3\times(1+2h)}] + (1+2h)^2 + 1] + ... + \{e^{2-3[1+(n-1)h]} + [1+(n-1)h]^2 + 1\}$$

$$\therefore \int_1^3 (e^{2-3x} + x^2 + 1)dx$$

$$= \lim_{h\to 0} h\,[e^2.(e^{-3} + e^{-3(1+h)} + e^{-3(1+2h)} + ... + e]$$
$$+ \lim_{h\to 0} h\,[1^2 + (1+h)^2 + (1+2h)^2 + ... + \{1 + (n-1)h\}^2]$$
$$+ \lim_{h\to 0} h(1 + 1 + ... n \text{ times})$$

$$= \lim_{h\to 0} h\left\{e^2 \times e^{-3}\frac{(1-e^{-3nh})}{1-e^{-3h}}\right\}$$
$$+ \lim_{h\to 0} h\{1 + 1 + ... n \text{ times}\} + 2h\,[1+2+3+...+(n-1)] + h^2(1^2+2^2+3^2+...+(n-1)^2)\}$$

$$+ \lim_{h\to 0} h(n) \left[\because \text{ sum to } n \text{ terms of G. P. } S_n = \frac{a(1-r^n)}{1-r}\right]$$

$$= \lim_{h\to 0} h\left\{\frac{e^{-1}(1-e^{-6})}{1-e^{-3h}}\right\} + \lim_{h\to 0} h\left[n+2h\times\frac{(n-1)n}{2} + h^2\times\frac{(n-1)n(2n-1)}{6}\right] + \lim_{h\to 0} h\,(n)$$

$$= \lim_{h\to 0} h\left(\frac{e^{-1}(1-e^{-6})}{1-e^{-3h}}\right) + \lim_{h\to 0}\left[2nh+(nh-h)\,nh+\frac{(nh-h)nh(2nh-h)}{6}\right]$$

$$= \frac{1}{e}\left(1-\frac{1}{e^6}\right)\times\frac{\lim_{h\to 0} e^{3h}}{3\times\lim_{h\to 0}\left(\frac{e^{3h}-1}{3h}\right)} + \lim_{h\to 0}\left[2\times 2+(2-h)\times 2+\frac{(2-h)\times 2(2\times 2-h)}{6}\right]$$

$$= \frac{1}{e}\left(1-\frac{1}{e^6}\right)\times\frac{1}{3\times 1}+\left(4+4+\frac{8}{3}\right)$$

$$= \frac{1}{3e}\left(1-\frac{1}{e^6}\right)+\frac{32}{3}$$ **Ans.**

22. Solve the differential equation :

$$(\tan^{-1} y - x)\, dy = (1+y^2)\, dx.$$

[6]

Solution : Same as solution Q. 23 (OR) Set 1 (Outside Delhi) upto eq.

$$x = \tan^{-1} y - 1 + c\, e^{\tan^{-1} y}$$

Ans.

OR

Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ given that $y = 1$, when $x = 0$.

Solution : Given, $$\frac{dy}{dx} = \frac{xy}{x^2+y^2} \quad ...(i)$$

which is a homogeneous differential equation.

Putting $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{vx^2}{x^2+v^2x^2}$$

$$x\frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$$

$$\Rightarrow \quad \frac{1+v^2}{v^3}dv = \frac{-1}{x}dx$$

$$\Rightarrow \quad \frac{1}{v^3}dv + \frac{1}{v}dv = \frac{-1}{x}dx$$

Now, integrating both sides, we get

$$\frac{v^{-3+1}}{-3+1} + \log|v| = -\log|x| + C$$

$$\Rightarrow \quad \frac{-1}{2v^2} + \log|v| = -\log|x| + C$$

$$\Rightarrow \quad \frac{-1}{2v^2} + \log|vx| = C$$

$$\Rightarrow \quad \frac{-x^2}{2y^2} + \log|y| = C \qquad (\because y = vx) ...(ii)$$

It is given that $y = 1$ when $x = 0$

Putting $x = 0, y = 1$ in (ii)

we get, $C = 0$

Putting $C = 0$ in (ii), we get

$$\log|y| = +\frac{x^2}{2y^2}$$

$$\Rightarrow \quad x^2 = +2y^2 \log|y|$$

Hence, $x^2 = 2y^2 \log|y|$ is the solution of the given equation. **Ans.**

23. If lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then find the value of k and hence find the equation of the plane containing these lines. [6]

Solution : The coordinates of any point on first line are

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = \lambda$$

i.e., $x = 2\lambda + 1,$
$y = 3\lambda - 1,$
$z = 4\lambda + 1$

i.e., $(2\lambda+1, 3\lambda-1, 4\lambda+1)$

and the coordinates of any point on second line are

$$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = \mu$$

i.e., $x = \mu + 3,$
$y = 2\mu + k,$
$z = \mu$

i.e., $(\mu+3, 2\mu+k, \mu)$

if these two lines intersect each other, then

$$2\lambda+1 = \mu+3,\ 3\lambda-1 = 2\mu+k,\ 4\lambda+1 = \mu$$

i.e. $2\lambda - \mu = 2,\ 3\lambda - 2\mu = k+1,\ 4\lambda - \mu = -1$

solving, $2\lambda - \mu = 2$ and $4\lambda - \mu = -1$, we get

$$\lambda = \frac{-3}{2} \text{ and } \mu = -5$$

and substituting the values of λ and μ in $3\lambda - 2\mu = k + 1$, we get

$$k = \frac{9}{2}$$

Now, we have $\vec{b_1} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b_2} = \hat{i} + 2\hat{j} + \hat{k}$. So, the required plane contains both lines and it passes through a point $\vec{a}\,(1, -1, 1)$ and perpendicular vector $\vec{n}$, given by, $\vec{n} = \vec{b_1} \times \vec{b_2}$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}(3-8) - \hat{j}(2-4) + \hat{k}(4-3)$$

$$= -5\hat{i} + 2\hat{j} + \hat{k}$$

$\therefore$ The equation of plane passing through $\vec{a}$ and perpendicular to $\vec{n}$ is given by

$$(\vec{r} - \vec{a}).\vec{n} = 0$$

$$[\vec{r} - (\hat{i} - \hat{j} + \hat{k})].(-5\hat{i} + 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow \vec{r}(-5\hat{i} + 2\hat{j} + \hat{k}) = (\hat{i} - \hat{j} + \hat{k}).(-5\hat{i} + 2\hat{j} + \hat{k})$$

$$\Rightarrow \vec{r}(-5\hat{i} + 2\hat{j} + \hat{k}) = -6$$

Writing $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

or $5x - 2y - z - 6 = 0.$ **Ans.**

24. If A and B are two independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(A \cap \bar{B}) = \frac{1}{6}$, then find P(A) and P(B). **[6]**

Solution : Let $P(A) = x$ and $P(B) = y$

$$\therefore \quad P(\bar{A}) = 1 - P(A) = 1 - x,\ P(\bar{B}) = 1 - y$$

We have, $P(\bar{A} \cap B) = \frac{2}{15}$ and $(P \cap \bar{B}) = \frac{1}{6}$

Now, $P(\bar{A} \cap B) = \frac{2}{15}$

$$\Rightarrow \quad P(\bar{A})\,P(B) = \frac{2}{15}$$

$$\Rightarrow \quad (1-x)y = \frac{2}{15} \quad \text{...(i)}$$

Since, A and B are independent events so are $\bar{A}$ and B as well as A and $\bar{B}$.

Given, $P(A \cap \bar{B}) = \frac{1}{6}$

$$\Rightarrow \quad P(A)P(\bar{B}) = \frac{1}{6}$$

$$\Rightarrow \quad x(1-y) = \frac{1}{6} \quad \text{...(ii)}$$

$$\Rightarrow \quad x = \frac{1}{6-6y}$$

Putting the value of x in eq. (i), we get

$$\left(1 - \frac{1}{6-6y}\right)y = \frac{2}{15}$$

$$\Rightarrow \quad -90y^2 + 87y = 12$$

or $30y^2 - 29y + 4 = 0$

Solving the quadratic equation, we get

$$y = \frac{4}{5}, y = \frac{1}{6}$$

For $y = \frac{4}{5}$; using (ii), we get

$$x = \frac{1}{6 - 6 \times \frac{4}{5}} = \frac{5}{30-24} = \frac{5}{6}$$

For $y = \frac{1}{6}$ using (ii), we get

$$x = \frac{1}{6 - 6 \times \frac{1}{6}} = \frac{1}{5}$$

$$\therefore \quad P(A) = \frac{5}{6},\ P(B) = \frac{4}{5}$$

or $P(A) = \frac{1}{5},\ P(B) = \frac{1}{6}.$ **Ans.**

25. Find the local maxima and local minima of the function $f(x) = \sin x - \cos x, 0 < x < 2\pi$. Also find the local maximum and local minimum values. **[6]**

Solution : We have, $f(x) = \sin x - \cos x, 0 < x < 2\pi$.

$$\Rightarrow \quad f'(x) = \cos x + \sin x$$

For local maximum or minimum, we have

$$\Rightarrow \quad f'(x) = 0$$

$$\Rightarrow \quad \cos x + \sin x = 0$$

$$\Rightarrow \quad \cos x = -\sin x$$

$$\Rightarrow \quad \tan x = -1$$

$$\Rightarrow \quad x = \frac{3\pi}{4} \text{ or } x = \frac{7\pi}{4}$$

Thus, $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ are possible points of local maximum or minimum.

We have, $f'(x) = \frac{d}{dx}(\cos x + \sin x)$

$= -\sin x + \cos x$

At $x = \frac{3\pi}{4}$, we have

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$$

$$\Rightarrow \quad f''\left(\frac{3\pi}{4}\right) < 0$$

So, $x = \frac{3\pi}{4}$ is the point of local maximum.

Local maximum value

$$= f\left(\frac{3\pi}{4}\right)$$

$$= \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

At $x = \frac{7\pi}{4}$, we have

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$$

$$\Rightarrow \quad f''\left(\frac{7\pi}{4}\right) > 0$$

So, $x = \frac{7\pi}{4}$, is the point of local minimum.

Local minimum value

$$= f\left(\frac{7\pi}{4}\right)$$

$$= \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}. \quad \textbf{Ans.}$$

26. Find graphically, the maximum value of $z = 2x + 5y$, subject to constraints given below : [6]

$$2x + 4y \leq 8$$
$$3x + y \leq 6$$
$$x + y \leq 4$$
$$x \geq 0, \; y \leq 0$$

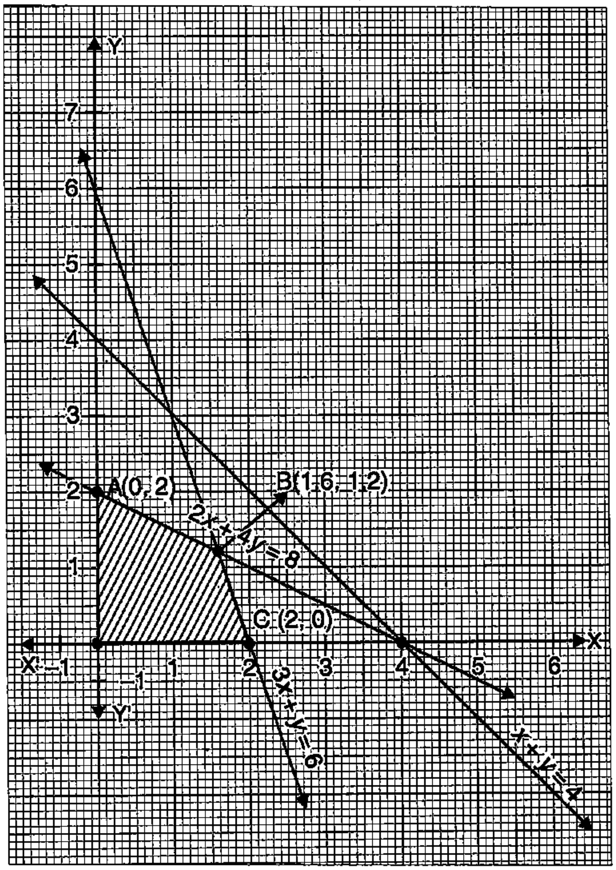
Solution : We first convert the inequalities into equations to obtain lines

$$2x + 4y = 8 \quad \text{...(i)}$$
$$3x + y = 6 \quad \text{...(ii)}$$
$$x + y = 4 \quad \text{...(iii)}$$
$$x = 0$$
and $$y = 0.$$

We need to maximize the objective function

$$z = 2x + 5y$$

These lines are drawn and the feasible region of the L.P.P. is the shaded region :



The point of intersection of (i) and (ii) is B (1.6, 1.2)

The coordinates of the corner points of the feasible region are O(0, 0), A(0, 2), B (1.6, 1.2) and C(2, 0).

The value of the objective function at these points are given in the following table :

Corner Points	Value of the objective function $z = 2x + 5y$
O(0, 0)	$2\times0+5\times0=0$
A(0, 2)	$2\times0+5\times2=10$ maximum
B(1.6, 1.2)	$2\times1.6+5\times1.2=9.2$
C(2, 0)	$2\times2+5\times0=4$

Out of these values of z, the maximum value of z is 10 which is attained at the point (0, 2). Thus the maximum value of z is 10. **Ans.**

●●

All questions are same in Delhi Set II and Set III